Lagrangian extensions of Lie superalgebras in characteristic 2

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Outline of the talk

1. Lie superalgebras and left-symmetric superalgebras in characteristic 2

2. Lagrangian extensions of Lie superalgebras

Lie superalgebras in characteristic 2, definition

A *Lie superalgebra* over a field $\mathbb{K}$ of characteristic $p = 2$ is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ such that:

1. the even part $\mathfrak{g}_{\bar{0}}$ is a Lie algebra;
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2. the odd part $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module ;
3. there is a map $s : \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$, satisfying $s(\lambda x) = \lambda^2 s(x)$, such that the bracket of two odd elements is given by:

$$[x, y] := s(x + y) - s(x) - s(y), \quad \forall x, y \in \mathfrak{g}_{\bar{1}}. \tag{1}$$

The Jacobi identity for two odds elements reads as follows:

$$[s(x), y] = [x, [x, y]], \ \forall x \in \mathfrak{g}_{\bar{1}}, \ \forall y \in \mathfrak{g}. \tag{2}$$

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Example: any associative superalgebra with $[a, b] = ab - ba$ and $s(a) = a^2$.

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- All the usual definitions have to take the squaring into account, for example the derived superalgebras of $\mathfrak{g}$ are given by

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] + \mathrm{Span}\{s(x),\ x \in (\mathfrak{g}^{(i)})_{\bar{1}}\}.$$

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- Lie superalgebras in characteristic 2 admitting a Cartan matrix have been classified by Bouarroudj, Grozman, Leites, SIGMA 2009.

Lie superalgebras, classification in dim $= 2$

Proposition

Let $\mathfrak{g}$ *be a 2-dimensional Lie superalgebras over an arbitrary field of characteristic* 2. *Then,* $\mathfrak{g}$ *is isomorphic to one of the following superalgebras.*

- <u>*sdim*$(\mathfrak{g}) = (0|2)$</u>: $\mathfrak{g} = \mathbf{L}^{1}_{0|2} = \langle 0|e_1, e_2\rangle$.
- <u>*sdim*$(\mathfrak{g}) = (1|1)$</u>; $\mathfrak{g} = \langle e_1|e_2\rangle$.
  1. $\mathbf{L}^{1}_{1|1} = \langle e_1|e_2; [e_1, e_2] = e_2\rangle$;
  2. $\mathbf{L}^{2}_{1|1} = \langle e_1|e_2; s(e_2) = e_1\rangle$;
  3. $\mathbf{L}^{3}_{1|1} = \langle e_1|e_2\rangle$ (*abelian*);
- <u>*sdim*$(\mathfrak{g}) = (2|0)$</u>: $\mathfrak{g} = \langle e_1, e_2|0\rangle$.
  1. $\mathbf{L}^{1}_{2|0} = \langle e_1, e_2|0\rangle\,; [e_1, e_2] = e_2$;
  2. $\mathbf{L}^{2}_{2|0} = \langle e_1, e_2|0\rangle$ (*abelian*);

Lie superalgebras in characteristic 2, cohomology (1)

This cohomology was introduced by Bouarroudj and Makhlouf (2023).

Let $\mathfrak{g}$ be a Lie superalgebra in characteristic 2 and let M be a $\mathfrak{g}$-module.

A 1-cocycle on $\mathfrak{g}$ with values in M is a linear map $\varphi : \mathfrak{g} \to M$ such that

$$\begin{aligned} d^1_{\mathrm{CE}}(\varphi)(x,z) &:= x\cdot\varphi(z) + z\cdot\varphi(x) + \varphi([x,z]) = 0, \forall x, z \in \mathfrak{g}; \\ \delta^1(\varphi)(x) &:= x\cdot\varphi(x) + \varphi(s(x)) = 0, \qquad\qquad \forall x \quad \in \mathfrak{g}_{\bar{1}}. \end{aligned}$$

The space of 1-cocycles on $\mathfrak{g}$ with values in M is denoted by $XZ^1(\mathfrak{g}; M)$.
We also use the notation $\mathfrak{d}^1(\varphi) := (d^1_{\mathrm{CE}}(\varphi), \delta^1(\varphi))$.

Lie superalgebras in characteristic 2, cohomology (2)

A 2-cocycle on $\mathfrak{g}$ with values in M consists of a pair (α, γ) such that:

1. $\alpha : \mathfrak{g} \wedge \mathfrak{g} \to M$ is a bilinear map;
2. $\gamma : \mathfrak{g}_{\bar{1}} \to M$ satisfies

$$\begin{aligned} \gamma(\lambda x) &= \lambda^2 \gamma(x), && \forall \lambda \in \mathbb{K},\ \forall x \in \mathfrak{g}; \\ \alpha(x, y) &= \gamma(x + y) + \gamma(x) + \gamma(y), && \forall x, y \in \mathfrak{g}_{\bar{1}}; \end{aligned}$$

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3. For all $x, y, z \in \mathfrak{g}$ and for all $u \in \mathfrak{g}_{\bar{1}}$, we have

$$d^2_{\mathrm{CE}}(\alpha)(x, y, z) := \underset{x,y,z}{\circlearrowleft} \left(x \cdot \alpha(y, z) + \alpha([x, y], z) \right) = 0;$$

$$\delta^2(\alpha, \gamma)(u, z) := u \cdot \alpha(u, z) + z \cdot \gamma(u) + \alpha(s(u), z) + \alpha([u, z], u) = 0.$$

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There is a graduation on $XZ^2(\mathfrak{g}; M)$ defined by $|(\alpha, \gamma)| := |\alpha|$.

Left-symmetric superalgebras in characteristic 2

A left-symmetric superalgebra $(V, \triangleright)$ in characteristic $p = 2$ is a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ endowed with a bilinear product $\triangleright : V \times V \to V$ satisfying

$$\begin{array}{lrcll} (i) & x \triangleright (y \triangleright z) + (x \triangleright y) \triangleright z & = & y \triangleright (x \triangleright z) + (y \triangleright x) \triangleright z, & \forall x, y, z \in V; \\ (ii) & x \triangleright (x \triangleright y) & = & (x \triangleright x) \triangleright y, & \forall x \in V_{\bar{1}},\ \forall y \in V. \end{array}$$

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Proposition

Let $(V, \triangleright)$ be a left-symmetric superalgebra. Then, $(\mathfrak{g}(V), [\cdot, \cdot], s)$ is a Lie superalgebra with $\mathfrak{g}(V) = V$ as superspaces and

$$[x, y] := x \triangleright y + y \triangleright x, \ \forall x \in V_{\bar 0}, \forall y \in V; \tag{3}$$

$$s(x) := x \triangleright x, \qquad \forall x \in V_{\bar 1}. \tag{4}$$

A left-symmetric product $\triangleright$ on a Lie superalgebra $(V, [\cdot, \cdot], s)$ is called compatible with the Lie superalgebra structure if Conditions (3) and (4) are satisfied.

Left-symmetric superalgebras in characteristic 2, example

Proposition

Let $(\mathfrak{g}, [\cdot,\cdot], s)$ be a Lie superalgebra equipped with an invertible derivation D. Let

$$x \triangleright y := D^{-1}\big([x, D(y)]\big), \ \forall x, y \in \mathfrak{g}.$$

Then, $\triangleright$ is a left-symmetric product compatible with the Lie structure.

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Example. Consider the Hamiltonian superalgebra $\mathfrak{h}_\Pi(0|4)$ (see Benayadi and Bouarroudj, Journal of Algebra, 2018). As a vector space it can be considered as

$$\mathfrak{h}_\Pi(0|4) \simeq \mathrm{Span}\{H_f \mid f \in \mathbb{K}[\xi,\eta]\} \simeq [\xi,\eta]/\,\mathbb{K}\cdot 1,$$

where $\xi_1, \xi_2, \eta_1, \eta_2$ are odd indeterminates and

$$H_f = \frac{\partial f}{\partial \xi_1}\frac{\partial}{\partial \eta_1} + \frac{\partial f}{\partial \eta_1}\frac{\partial}{\partial \xi_1} + \frac{\partial f}{\partial \xi_2}\frac{\partial}{\partial \eta_2} + \frac{\partial f}{\partial \eta_2}\frac{\partial}{\partial \xi_2}.$$

The Lie bracket $[H_f, H_g] = H_{\{f,g\}}$ is given by the Poisson bracket:

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Then, its $\boxed{\text{simple}}$ derived superalgebra $\mathfrak{h}_\Pi^{(1)}(0|4)$ admits an invertible derivation, thus a left-symmetric structure.

Classification in dimension 2

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		$e_1 e_1 = e_1;\ e_1 e_2 = e_2$	None
		$e_1 e_1 = \varepsilon e_1;\ e_1 e_2 = (1+\varepsilon) e_2;\ e_2 e_1 = \varepsilon e_2$	$\varepsilon \neq 0$
$\mathbf{L}^2_{1\|1}$	$s(e_2) = e_1$	$e_2 e_2 = e_1$	None
		$e_1 e_1 = e_1;\ e_1 e_2 = e_2;\ e_2 e_1 = e_2;\ e_2 e_2 = e_1$	None
$\mathbf{L}^3_{1\|1}$	abelian	$e_1 e_1 = e_1$	None
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The case where $\mathrm{sdim}(\mathfrak{g}) = (1|1)$.

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The case where $\mathrm{sdim}(\mathfrak{g}) = (1|1)$.

- $\mathfrak{g} = \mathbf{L}^1_{2|0}$, $[e_1, e_2] = e_2$: 10 non-isomorphic left-symmetric products;
- $\mathfrak{g} = \mathbf{L}^2_{2|0}$, abelian: 5 non-isomorphic left-symmetric products.

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The **torsion of the connection** ∇ is given by a pair of maps (T, U), where $T : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$ and $U : \mathfrak{g}_{\bar{1}} \to \mathfrak{g},$ are defined by

$$\begin{aligned} T(x,y) &:= \nabla_x(y) + \nabla_y(x) + [x,y], & \forall x, y \in \mathfrak{g}; \\ U(x) &:= \nabla_x(x) + s(x), & \forall x \in \mathfrak{g}_{\bar{1}}. \end{aligned}$$

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The **curvature of the connection** ∇ is given by a pair of maps (R, S), where $R : \mathfrak{g} \times \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, and $S : \mathfrak{g}_{\bar{1}} \to \mathrm{End}(\mathfrak{g})$, are defined by

$$\begin{aligned} R(x,y) &:= \nabla_{[x,y]} + [\nabla_x, \nabla_y], && \forall x, y \in \mathfrak{g}; \\ S(x) &:= \nabla_{s(x)} + \nabla_x^2, && \forall x \in \mathfrak{g}_{\bar{1}}. \end{aligned}$$

The connection ∇ is called *flat* if $(R, S) = (0, 0)$.

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The operation $\triangleright$ is a left-symmetric product compatible with the bracket and the squaring of $\mathfrak{g}$ if and only if the connection ∇ is flat and torsion-free.

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As in characteristic zero, a flat connection on $\mathfrak{g}$ whose covariant derivative of the torsion vanishes defines a (characteristic 2 version) of a **post-Lie** product.

Lagrangian extensions

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- The superization is due to Bouarroudj and Maeda (J. Algebra Appl., 2023)

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- To our best knowledge, Lagrangian extensions were introduced by M. Bordemann under the name *T^*-extensions*, dealing with symmetric bilinear forms (Acta Math. Univ. Comenianae, 1997).

- Roughly speaking, starting from a non-associative algebra A, he defines a product and a symmetric bilinear form on the space $A \oplus A^*$.

- The notion was then studied by Baues and Cortes, who name it *Lagrangian extensions*, in the context of Lie algebras admitting a flat connection and the bilinear form is antisymmetric (Astérisque, 2016).

- The superization is due to Bouarroudj and Maeda (J. Algebra Appl., 2023)

- **Our goal:** the case of Lie superalgebras in characteristic 2 (with Benayadi and Bouarroudj, to appear in Journal of Pure and Applied Algebra)

Quasi-Frobenius Lie superalgebras

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra in characteristic 2. A bilinear form ω on $\mathfrak{g}$ with values in $\mathbb{K}$ is called

1. *ortho-orthogonal* if ω is even;
2. *periplectic* if ω is odd;
3. *closed* if the following cocycle conditions are satisfied:

$$\underset{x,y,z}{\circlearrowleft} \omega([x,y],z) = 0, \qquad \forall x,y,z \in \mathfrak{g}; \tag{5}$$

$$\omega(s(x),y) = \omega(x,[x,y]), \ \forall x \in \mathfrak{g}_{\bar{1}}, \ \forall y \in \mathfrak{g}. \tag{6}$$

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An $\boxed{\textit{even}}$ bilinear form on $\mathfrak{g}$ is called $\bar{1}$-*antisymmetric* if

$$\omega(x,y) = \omega(y,x), \ \forall x,y \in \mathfrak{g} \text{ s. t. } |x| = |y|; \text{ and } \omega(x,x) = 0 \ \forall x \in \mathfrak{g}_{\bar{0}}.$$

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A Lie superalgebra $\mathfrak{g}$ is called *quasi-Frobenius* if it is equipped with a $\bar{1}$-antisymmetric non-degenerate closed form ω.

Construction of the Lagrangian extensions

In the sequel, I will focus on the $\boxed{\text{ortho-orthogonal}}$ case.

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Let $(\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}}, s_{\mathfrak{h}})$ be a Lie superalgebra endowed with a torsion-free flat connection $\nabla : \mathfrak{h} \to \mathrm{End}(\mathfrak{h})$. We define the *dual representation*

$$\rho : \mathfrak{h} \to \mathrm{End}(\mathfrak{h}^*), \quad \rho(x)(\xi) := \xi \circ \nabla_x.$$

Let $(\alpha, \gamma) \in XZ^2(\mathfrak{h}, \mathfrak{h}^*)_{\bar{0}}$ be a 2-cocycle.

On the space $\mathfrak{g} := \mathfrak{h} \oplus \mathfrak{h}^*$. The brackets and squaring are defined as follows:

$$[x, y]_{\mathfrak{g}} := [x, y]_{\mathfrak{h}} + \alpha(x, y), \quad [x, \xi]_{\mathfrak{g}} := \rho(x)(\xi), \ \forall x, y \in \mathfrak{h}, \ \forall \xi \in \mathfrak{h}^*.$$

$$s_{\mathfrak{g}}(x + \xi) := s_{\mathfrak{h}}(x) + \gamma(x) + \rho(x)(\xi), \quad \forall x \in \mathfrak{h}_{\bar{1}}, \ \forall \xi \in \mathfrak{h}^*_{\bar{1}}.$$

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We define an $\boxed{\text{ortho-orthogonal}}$ form as follows:

$$\omega(x + \xi, y + \zeta) = \xi(y) + \zeta(x), \quad \forall x + \xi, y + \zeta \in \mathfrak{g}.$$

Construction of the Lagrangian extensions

We define the first and second Lagrangian cochain spaces as

$$XC^1_L(\mathfrak{h},\mathfrak{h}^*) := \big\{\psi \in XC^1(\mathfrak{h},\mathfrak{h}^*),\ \psi(x)(y) = \psi(y)(x),\ \forall x, y \in \mathfrak{h}\big\},$$
$$XC^2_L(\mathfrak{h},\mathfrak{h}^*) := \big\{(\alpha,\gamma) \in XC^2(\mathfrak{h},\mathfrak{h}^*),\ \text{satisfying (7) and (8)}\big\},\ \text{where}$$

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Theorem

Let $(\mathfrak{h},\nabla)$ be a Lie superalgebra equipped with a flat and torsion-free connection ∇ and let $(\alpha,\gamma) \in XZ^2(\mathfrak{h},\mathfrak{h}^*)_{\bar{0}}$ be an even 2-cocycle.

1. *The form ω on $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^*$ is closed if and only if (α,γ) is a $\boxed{\text{Lagrangian}}$ 2-cocycle.*

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2. *In this case, one can canonically define a strongly polarized quasi-Frobenius Lie superalgebra $(\mathfrak{g},\omega,\mathfrak{h}^*,\mathfrak{h})$, where $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^*$, called T^*-extension of $(\mathfrak{h},\nabla)$.*

The converse

Let $(\mathfrak{g}, \omega)$ be a quasi-Frobenius Lie superalgebra. A **strong polarization** of $(\mathfrak{g}, \omega)$ is a decomposition $\mathfrak{g} = \mathfrak{a} \oplus N$ as vector superspaces, where $\mathfrak{a}$ is a homogeneous Lagrangian ideal of $\mathfrak{g}$ ($\mathfrak{a}^{\perp} = \mathfrak{a}$) and N is a Lagrangian subspace.

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Let $(\mathfrak{g}, \omega_{\mathfrak{g}}, \mathfrak{a}, N)$ be a strongly polarized quasi-Frobenius Lie superalgebra and let $\mathfrak{h} := \mathfrak{g} / \mathfrak{a}$.

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Theorem

Let $(\mathfrak{g}, \omega_{\mathfrak{g}}, \mathfrak{a}, N)$ be a strongly polarized quasi-Frobenius Lie superalgebra and let $\mathfrak{h} := \mathfrak{g}/\mathfrak{a}$.
If ω is ortho-orthogonal, there exists an even Lagrangian cocycle (α, γ) and a a flat and torsion-free connection ∇ on $\mathfrak{h}$ such that $(\mathfrak{g}, \omega, \mathfrak{a}, N)$ is isomorphic to the T^*-extension of $(\mathfrak{h}, \nabla)$ by (α, γ).

Equivalence of Lagrangian extensions

An *isomorphism of Lagrangian extensions of* $\mathfrak{h}$ is a Lie isomorphism $\Phi : (\mathfrak{g}, \omega) \to (\mathfrak{g}', \omega')$ satisfying $\omega(x, y) = \omega'\big(\Phi(x), \Phi(y)\big),\ \forall x, y \in \mathfrak{g}$, such that the following diagram commutes:

$$
\begin{tikzcd}
0 \arrow[r] & \mathfrak{a} \arrow[r] \arrow[d, "\Phi|_{\mathfrak{a}}"'] & \mathfrak{g} \arrow[r] \arrow[d, dashed, "\Phi"] & \mathfrak{h} \arrow[r] \arrow[d, "="] & 0 \\
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\end{tikzcd}$$

Theorem

Let $(\mathfrak{h}, \nabla)$ be a Lie superalgebra equipped with a flat torsion-free connection ∇. Two Lagrangian extensions of $(\mathfrak{h}, \nabla)$ are isomorphic if and only if they correspond to the same extension 2-cocycle in $XH^2_L(\mathfrak{h}, \mathfrak{h}^*)_{\bar{0}}$.

An example

Consider $\mathfrak{h} := \mathbf{L}^{\mathbf{1}}_{\mathbf{1}|\mathbf{1}}$ given in the basis $(e|f)$ by $[e,f] = f$ and $s = 0$. Let $\varepsilon \in \mathbb{K}$. We define a flat torsion-free connection ∇^{ε} on $\mathfrak{h}$ by

$$\nabla^{\varepsilon}_e(e) = (1+\varepsilon)e, \quad \nabla^{\varepsilon}_e(f) = \varepsilon f, \quad \nabla^{\varepsilon}_f(e) = (1+\varepsilon)f, \quad \nabla^{\varepsilon}_f(f) = 0.$$

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1. In the case where $\varepsilon \neq 1$, we have $XH^2(\mathfrak{h}, \mathfrak{h}^*) = 0$;
2. In the case where $\varepsilon = 1$, $XH^2_L(\mathfrak{h}, \mathfrak{h}^*)$ is one-dimensional and spanned by (α_2, γ^1_3), where
$$\alpha_2 = f^* \otimes e^* \wedge f^*; \quad \gamma^1_3(f) = e^*.$$

Thank you for your attention

Main reference:

S. Benayadi, S. Bouarroudj, Q. Ehret,
Left-symmetric superalgebras and Lagrangian extensions of Lie superalgebras in characteristic 2,
`arXiv:2501.15432`,
to appear in Journal of Pure and Applied Algebra.