Problems 1

1. Let G be a group. Show that the inverse of an element is unique.
2. Let G be a group and let $a, b, c \in G$. Show that $ab = ac \Rightarrow b = c$.
3. For $n \geq 1$, consider
$$SL_n(\mathbb{C}) = \{M \in GL_n(\mathbb{C}),\ \det(M) = 1\}\,.$$
Show that $SL_n(\mathbb{C})$ is a subgroup of $GL_n(\mathbb{C})$.
4. Consider the following sets equipped with a composition law. Determine whether they are a group or not.
   (a) $(\mathbb{R}, *)$ with $x * y = x + y - 1,\ \forall x, y \in \mathbb{R}$;
   (b) $\{f : \mathbb{R} \to \mathbb{R}, \circ\}$ (composition);
   (c) For $n \geq 2$, $\{A \in M_n(\mathbb{Z}), \det(A) \neq 0, \cdot)\}$ (matrix product);
   (d) For $n \geq 2$, $\{A \in M_n(\mathbb{Z}), \exists B \in M_n(\mathbb{Z}), AB = Ba = I_n, \cdot)\}$ (matrix product);
5. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 6 & 1 & 4 \end{pmatrix} \in S_6$. Compute σ^3. Same question with $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix}$.
6. For $n \geq 2$, Show that S_n is generated by the $n-1$ transpositions $(12), (13), \cdots, (1n)$.
7. Let λ, μ be real numbers. Define a binary operation $*$ on $\mathbb{R}$ by
$$a * b := \lambda ab + \mu(a + b).$$
Find conditions on λ, μ such that $*$ is associative.
8. (Later) Let $n \geq 3$ and consider $A_n = \{\sigma \in S_n,\ sgn(\sigma) = +1\}$. Show that A_n is a subgroup of S_n. Show that A_n is generated by the 3-cycles.
9. Let $n \geq 2$. Show that $n\mathbb{Z}$ is a subgroup of $\mathbb{Z}$.