Problems 2

1. Let $H \subset G$ be a subgroup of G. Show that the following assertions are equivalent :
   (i) H is a normal subgroup of G ;
   (ii) $H = gHg^{-1},\ \forall g \in G$;
   (iii) $ghg^{-1} \in H,\ \forall g \in G,\ \forall h \in H$.

2. Let $\varphi : G \to H$ be a group homomorphism.
   (a) Show that $\mathrm{Im}(\varphi)$ is a subgroup of H.
   (b) Show that $\ker(\varphi)$ is a normal subgroup of G.
   (c) Show that $\ker(\varphi) = 1_G \iff \varphi$ is injective.

3. Decide whether the following maps are morphisms of groups. If yes, compute their kernel and image. Decide whether they are isomorphisms or not.
   (a) $f : M_n(\mathbb{R}, +) \to M_n(\mathbb{R}, +),\ A \mapsto A + A^t$;
   (b) $f : GL_n(\mathbb{R}) \to GL_n\mathbb{R}),\ A \mapsto A^t$;
   (c) $\det : GL_n(\mathbb{R}) \mapsto \mathbb{R}^*$;
   (d) $f : \mathbb{C}^* \mapsto \mathbb{R}^*,\ z \mapsto |z|$.

4. Let $H \subset G$ such that $[G : H] = 2$. Show that H is a normal subgroup of G.

5. Let $n \geq 3$. Show that $Z(S_n) = \{\mathrm{id}\}$.

6. Let $(G, *)$ be a finite group and let $\varphi : (G, *) \to \mathbb{C}^*$ be a non-constant morphism of groups. Show that
$$\sum_{x \in G} \varphi(x) = 0.$$