Problems 6

1. For $n \geq 1$, the Dihedral group is defined by

$$D_n = \{r^i s^j,\ r^n = s^2 = (rs)^2 = 1\}.$$

   (a) Recall/explain the geometrical intrepretation of D_n as the symmetry group of a regular planar n-gon.

   (b) Show that D_6 is isomorphic to $D_3 \times C_2$, where C_2 is the cyclic group of order 2. *Hint : use Artin's Prop. 2.11.4.*

2. Let G be a group and X be a set. The group G operates on X if there is a map

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto gx, \end{aligned}$$

such that $1x = x$ and $g(hx) = (gh)x, \quad \forall g, h \in G,\ \forall x \in X$. Moreover, for all $x \in X$, we define

$$\mathrm{Stab}(x) = \{g \in G,\ gx = x\}, \quad \mathrm{Orb}(x) = \{y \in X,\ \exists g \in G,\ y = gx\}.$$

Show the following counting formula : $|G| = |\mathrm{Stab}(x)| \times |\mathrm{Orb}(x)|, \quad \forall x \in X.$

3. In this question, we will investigate the group of symmetries of a regular tetrahedron $\mathcal{T}$ in $\mathbb{R}^3$, that is, the group of transformations of the space that leave the regular tetrahedron invariant.

   (a) Denote by G the set of symmetries of $\mathcal{T}$. Show that any $\varphi \in G$ leave the set of vertices of $\mathcal{T}$ invariant.

   (b) Show that G is a group equipped with a homomorphism $\chi : G \to S_4$.

   (c) Show that χ is a group isomorphism.

4. In this question, we will investigate the group of symmetries of a cube $\mathcal{C}$ in $\mathbb{R}^3$, that is, the group of transformations of the space that leave the cube invariant. Denote this group by G. The group G is acting on the cube $\mathcal{C}$.

   (a) Let A be a vertex of the cube. Compute $\mathrm{Stab}(A)$ and $\mathrm{Orb}(A)$. Deduce that $|G| = 48$.

   (b) Let $\mathcal{D}$ be the set of big diagonals of the cube, that is, $\mathcal{D} = \{(IJ),\ I, J \in \mathcal{C},\ |I - J| = \sqrt{3}\}$. Denote by $S(\mathcal{D})$ the group of symmetries of $\mathcal{D}$. Show that the morphism

$$\psi : G \to S(\mathcal{D}),\ f \mapsto f|_{\mathcal{D}}$$

   is surjective, but not injective. Show that $\ker(\psi) \cong \mathbb{Z}/2\mathbb{Z}$.

   (c) Consider the determinant $\det : G \to \{\pm 1\}$. Explain why it is a surjective group morphism. Denote by $G^+ = \ker(\det)$. Show that $|G^+| = |S_4|$ and deduce that $G^+ \cong S_4$.

   (d) Consider the map

$$\begin{aligned} \chi : G^+ \times \ker(\psi) &\to G, \\ (\sigma, \tau) &\mapsto \sigma \circ \tau. \end{aligned}$$

   Show that χ is an isomorphism of groups. Deduce that $G \cong S_4 \times \mathbb{Z}/2\mathbb{Z}$.