The superization of Hochschild's Lemma and restricted Lie-Rinehart superalgebras

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Lie Theory and Related Areas
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Joint work with
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Outline of the talk

1. Lie-Rinehart (super)algebras
2. Restricted Lie algebras, restricted Lie-Rinehart algebras
3. The superization of Hochschild's Lemma
4. Modules, semi-direct product, universal enveloping algebra

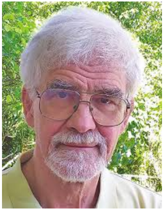
Lie-Rinehart algebras: definition

J. Herz (1953) - "pseudo-algèbre de Lie";

R. Palais (1961) - "d-Lie ring";

G. Rinehart (1963) - "(R, A)-Lie algebras";
Differential form on general commutative algebras, Trans. AMS **108**

J. Huebschmann (1990) - "Lie-Rinehart algebra".
Poisson cohomology and quantization, J. Reine Angew. Math. **408**



Johannes Huebschmann

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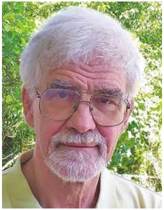

Johannes Huebschmann

Definition

A **Lie-Rinehart algebra** is a triple (A, L, ρ), where

- A is an associative commutative algebra;
- $(L, [-,-])$ is a Lie algebra that is also an A-module;
- $\rho : L \to \mathrm{Der}(A)$ is an A-linear Lie algebras morphism satisfying the *Leibniz identity*

$$[x, ay] = a[x, y] + \rho(x)(a)y, \quad \forall a \in A, \ \forall x, y \in L.$$

Lie-Rinehart algebras: examples

- Let A be an associative commutative algebra. Then, $\mathrm{Der}(A)$ is a Lie algebra with the commutator bracket and the triple $(A, \mathrm{Der}(A), \mathrm{id})$ is a Lie-Rinehart algebra.

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- (Huebschmann) Let $(A, \{-,-\})$ be a Poisson algebra with unit. Let $\Omega(A)$ be its module of Kähler differentials:

$$\Omega(A) = \Big\{ da,\ a \in A,\ d(a+b) = da + db,\ d(ab) = dab + adb,\ d1 = 0 \Big\}.$$

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Then $(A, \Omega(A), \rho)$ is a Lie-Rinehart algebra.

- **Universal enveloping algebra** (Rinehart 1963, Huebschmann 1990).

$$U(A,L,\rho) = U^{+}(A \rtimes L)/ < i(a)i(b+x) - i(a(b+x)),\ a,b \in A,\ x \in L >$$

Lie-Rinehart *super*algebras

S. Chemla (*Manuscripta Math.* **87**, 1995),

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A **restricted Lie algebra** is a Lie algebra L equipped with a map $(-)^{[p]} : L \longrightarrow L$ satisfying for all $x, y \in L$ and for all $\lambda \in \mathbb{K}$:

1. $(\lambda x)^{[p]} = \lambda^p x^{[p]}$;
2. $\left[x, y^{[p]}\right] = [[\cdots[x, \overbrace{y], y], \cdots, y}^{p \text{ terms}}]$;
3. $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$,

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with $is_i(x, y)$ the coefficient of Z^{i-1} in $\mathrm{ad}^{p-1}_{Zx+y}(x)$. Such a map $(-)^{[p]} : L \longrightarrow L$ is called p-map.

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Example: any associative algebra A with $[a, b] = ab - ba$ and $a^{[p]} = a^p$, $\forall a, b \in A$.

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Theorem (Jacobson)

Let L be a Lie algebra. Let $(e_j)_{j\in J}$ be a basis of L, and let the elements $f_j \in L$ be such that $(\mathrm{ad}_{e_j})^p = \mathrm{ad}_{f_j}$. Then, there exists exactly one p-mapping $(-)^{[p]} : L \to L$ such that

$$e_j^{[p]} = f_j \quad \text{for all } j \in J.$$

Towards restricted Lie-Rinehart algebras

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Hochschild's Lemma

- U : associative algebra over the field of integers modulo p
- $V \subset U$: commutative subalgebra
- $D_u(w) = uw - wu, \ \forall u, w \in U$.

Then, for all $u \in U$ such that $D_u(V) \subset V$, we have

$$(vu)^p = v^p u^p + D_{vu}^{p-1}(v)u, \ \forall v \in V.$$

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Let (A, L, ρ) be a Lie-Rinehart algebra. Applying the Lemma with U the universal enveloping algebra of (A, L, ρ), and $V = A$, we obtain

$$(ax)^p = a^p x^p + \rho(ax)^{p-1}(a)x, \quad \forall a \in A, \ \forall x \in L.$$

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- "Modern version", P. Schauenburg (2016).
  Let (A, L, ρ) be a Lie-Rinehart algebra and let (ϕ, M) be a Lie-Rinehart module. Then, we have

$$\phi(ax)^p = a^p \phi(x)^p + \rho(ax)^{p-1}(a)\phi(x).$$

Restricted Lie-Rinehart algebra: definition

- D. Rumynin, *Duality for Hopf algebroids*, J. Algebra **223** (2000);
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A Lie-Rinehart algebra (A, L, ρ) is called *restricted* if L is a restricted with a p-map $(-)^{[p]} : L \to L$, and if moreover, we have

$$(ax)^{[p]} = a^p x^{[p]} + \rho(ax)^{p-1}(a)x, \quad \forall a \in A, \ \forall x \in L. \tag{1}$$

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Our leading example (Dokas). Let A be an associative commutative algebra. Then, $(A, \mathrm{Der}(A), \mathrm{id})$ is a restricted Lie-Rinehart algebra.

Proof: Hochschild's Lemma.

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- **Goal:** to define the notion of restricted Lie-Rinehart superalgebra;
- For any associative supercommutative superalgebra A, the triple $(A, \mathrm{Der}(A), \mathrm{id})$ must provide an example.
- **Idea:** to prove a super-version of the Hochschild's Lemma.
- **Strategy:** generalize the proof of Schauenburg to the supercase.

Superization of Hochschild's Lemma

Theorem (Bouarroudj, E., Makhlouf, Shyntas)

Let (A, L, ρ) be a Lie-Rinehart superalgebra over a field $\mathbb{K}$ of characteristic $p \geq 3$ and let (ϕ, M) be a Lie-Rinehart module. Then, we have

$$\phi(ax)^p = a^p\phi(x)^p + \rho(ax)^{p-1}(a)\phi(x), \qquad \forall a \in A_{\bar{0}},\ \forall x \in L_{\bar{0}}; \tag{2}$$

$$\begin{aligned}\phi(ax)^{2p} &= a^{2p}\phi(x)^{2p} + \rho(ax)^{2p-1}(a)\phi(x) \\ &\quad + \sum_{i=0}^{p-1} \lambda_i \rho(ax)^i(a)\rho(ax)^{2p-2-i}(a)\phi(x)^2, \qquad \forall a \in A_{\bar{0}},\ \forall x \in L_{\bar{1}};\end{aligned} \tag{3}$$

$$\phi(ax)^{2p} = 0, \qquad \forall a \in A_{\bar{1}},\ \forall x \in L_{\bar{0}}; \tag{4}$$

$$\phi(ax)^p = a\big(\rho(x)(a)\big)^{p-1}\phi(x), \qquad \forall a \in A_{\bar{1}},\ \forall x \in L_{\bar{1}}, \tag{5}$$

where the coefficients λ_i are given by

$$\lambda_i = \begin{cases} 2(-1)^{\frac{i}{2}} & \text{if } i \text{ is even, } 0 \leq i < p-1; \\ 2(-1)^{\frac{i-1}{2}} & \text{if } i \text{ is odd, } 1 \leq i < p-1; \\ (-1)^{\frac{p-1}{2}} & \text{if } i = p-1. \end{cases} \tag{6}$$

Superization of Hochschild's Lemma

Elements of the proof.

Consider the $\mathbb{Z}_2$-graded ring $V = \mathbb{Z}[x_0, x_1, x_2, \cdots]$, such that $|x_{i+1}| = |x_i| + |\delta|$, where δ is the derivation of V defined by $\delta(x_i) = x_{i+1}$.

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Key point: study the polynomials $\Gamma_{k,j}$ defined by $\Gamma_{1,1} = x_0$ and

$$\Gamma_{k+1,j} = \begin{cases} x_0\delta(\Gamma_{k,1}), & j = 1 \\ x_0\delta(\Gamma_{k,j}) + (-1)^{|\delta||\Gamma_{k,j-1}|}x_0\Gamma_{k,j-1}, & j = 2, \cdots, k \\ x_0\Gamma_{k,k}, & j = k+1. \end{cases}$$

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Proposition (Case x_0 even, δ odd)

$$\Gamma_{2p,j} \equiv 0 \mod p, \quad \textit{for all} \quad 3 \leq j \leq 2p-1.$$

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Let (A, L, ρ) be a Lie-Rinehart superalgebra. Applying a well-chosen map $f : V \to A$ gives the result. $\square$

Restricted Lie-Rinehart superalgebras

Definition (Restricted Lie superalgebra)

A **restricted Lie superalgebra** is a Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ such that

1. The even part $L_{\bar{0}}$ is a restricted Lie algebra;
2. The odd part $L_{\bar{1}}$ is a Lie $L_{\bar{0}}$-module;
3. $\left[x, y^{[p]}\right] = [[\ldots[x, \overbrace{y], y], \ldots, y}^{p \text{ terms}}], \ \forall x \in L_{\bar{1}}, \ y \in L_{\bar{0}}.$

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We can define a map $(\cdot)^{[2p]} : L_{\bar{1}} \to L_{\bar{0}}$ by

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Examples.

- associative superalgebras;
- $\mathrm{Der}(A)$ with A associative superalgebra.

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A Lie-Rinehart superalgebra (A, L, ρ) is called *restricted* if the Lie superalgebra L is restricted and if in addition, we have

$$(ax)^{[p]} = a^p x^{[p]} + \rho(ax)^{p-1}(a)x, \qquad \forall a \in A_{\bar{0}},\ \forall x \in L_{\bar{0}}; \tag{7}$$

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Case $p = 3$. Let $a \in A_{\bar{0}}$ and $D \in \mathrm{Der}(A)_{\bar{1}}$.

$$(aD)^6 = a^6 D^6 + (aD)^5(a)D + 2a^4 D^3(a)D(a)D^2 + 2a^5 D^4(a)D^2.$$

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However, one can show that

$$\begin{aligned} 2a(aD)^4(a)D^2 &= a^4 D^2(a)^2 D^2 + 2a^4 D(a)D^3(a)D^2 + 2a^5 D^4(a)D^2; \\ 2(aD)(a)(aD)^3(a)D^2 &= a^4 D^3(a)D(a)D^2; \\ 2(aD)^2(a)^2 D^2 &= 2a^4 D^2(a)^2 D^2. \end{aligned}$$

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It follows that

$$(aD)^6 = a^6 D^6 + (aD)^5(a)D + 2a(aD)^4(a)D^2 + 2(aD)(a)(aD)^3(a)D^2 + 2(aD)^2(a)^2 D^2.$$

We therefore recover Eq. (8).

Modules and representations

Let (A, L, ρ) be a restricted Lie-Rinehart superalgebra. A representation of (A, L, ρ) is an A-module $V = V_{\bar{0}} \oplus V_{\bar{1}}$ together with a A-linear morphism of restricted Lie superalgebras $\phi : L \to \mathrm{End}(V)$ satisfying:

$$
\begin{aligned}
\phi(x)(av) &= (-1)^{|x||a|} a\phi(x)(v) + \rho(x)(a)v, && \forall x \in L,\ \forall a \in A,\ \forall v \in V,\\
\phi(ax)^{p-1}(av) &= a^p \phi(x)^{p-1}(v) + \rho(ax)^{p-1}(a)v, && \forall x \in L_{\bar{0}},\ \forall a \in A_{\bar{0}},\ \forall v \in V,\\
\phi(ax)^{2p-1}(av) &= a^{2p}\phi(x)^{2p-1}(v) + \rho(ax)^{2p-1}(a)v \\
&\quad + \sum_{i=0}^{p-1} \lambda_i \rho(ax)^i(a)\rho(ax)^{2p-2-i}(a)\phi(x)(v), && \forall a \in A_{\bar{0}},\ \forall x \in L_{\bar{1}}\ \forall v \in V,
\end{aligned}
$$

where the coefficients λ_i are defined as before. Such a pair (ϕ, V) is called a *restricted Lie-Rinehart module*.

Semi-direct product

Proposition

Let $\left(L, [-,-], (-)^{[p|2p]}\right)$ be a restricted Lie superalgebra and let (ϕ, V) be a restricted representation. Then, $L \rtimes V$ is a restricted Lie superalgebra with the bracket

$$\left[(x+v),(y+w)\right]_{\rtimes} := [x,y] + \phi(x)(w) - (-1)^{|y||v|}\phi(y)(v), \quad \forall\, x,y \in L, \ \forall\, v,w \in V$$

and with a $p|2p$-map $(-)^{[p|2p]_{\rtimes}} : L \rtimes V \to (L \rtimes V)_{\bar{0}}$ satisfying

$$(e_i + v_j)^{[p]_{\rtimes}} = e_i^{[p]} + \phi(e_i)^{p-1}(v_j),$$

where $(e_i)_i$ forms a basis of $L_{\bar{0}}$ and $(v_j)_j$ a basis of $V_{\bar{0}}$.

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Theorem

Let (A, L, ρ) be a restricted Lie superalgebra and let (ϕ, V) a representation. Suppose that the center of the restricted Lie superalgebra $L \ltimes V$ is trivial. Then, $(A, L \ltimes V, \widetilde{\rho})$ is a restricted Lie-Rinehart superalgebra, with $\widetilde{\rho}(x+v) = \rho(x)$, $\forall\, x+v \in L \ltimes V$.

The universal enveloping algebra, ordinary case

Let (A, L, ρ) be a Lie-Rinehart superalgebra. The superspace $A \oplus L$, is a Lie superalgebra with the bracket

$$[a + x, b + y] = [x, y] + \rho(x)(b) - (-1)^{|a||y|}\rho(y)(a), \quad \forall a + x, b + y \in A \oplus L.$$

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Denote by $U^+(A \rtimes L)$ the subspace of $U(A \rtimes L)$ spanned by $i(A \rtimes L)$ where $i : A \rtimes L \to U(A \rtimes L)$ is the (even) injection.

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The universal enveloping algebra of the Lie-Rinehart superalgebra (A, L, ρ) is defined by

$$U(A, L) = U^+(A \rtimes L)/J,$$

where

$$J = \left\langle i(a)i(b + x) - i(a(b + x)),\ a, b \in A,\ x \in L \right\rangle.$$

The universal enveloping algebra, restricted case

Let (A, L, ρ) be a *restricted* Lie-Rinehart superalgebra. Recall that $U(A, L) = U^+(A \rtimes L)/J$, where

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For all $a \in A$ and all $x \in L$, we have

$$i_A(a)i_L(x) = i_L(ax); \quad i_A\big(\rho(x)(a)\big) = i_L(x)i_A(a) - (-1)^{|a||x|} i_A(a)i_L(x).$$

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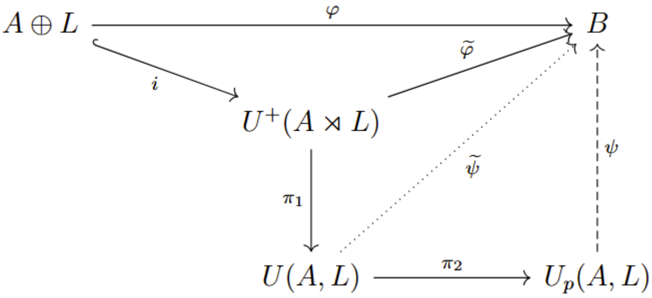

The universal enveloping algebra, restricted case

Universal property

- B an associative superalgebra,
- $j_A : A \to B$ an even morphism of associative superalgebras,
- $j_L : L \to B$ an even morphism of restricted Lie superalgebras,

satisfying for all $a \in A$ and all $x \in L$ the conditions

$$j_L(ax) = j_A(a) j_L(x); \quad \text{and} \quad j_A\big(\rho(x)(a)\big) = j_L(x) j_A(a) - (-1)^{|a||x|} j_A(a) j_L(x).$$

Then, there exists a unique morphism of associative superalgebras $\psi : U_p(A, L) \to B$ such that $\psi \circ i_L = j_L$ and $\psi \circ i_A = j_A$.

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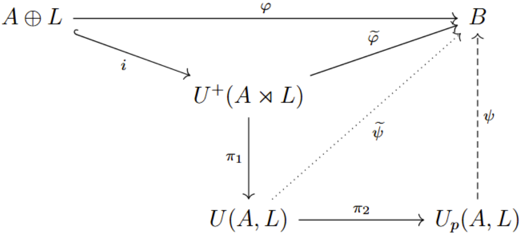

Thank you for your attention

Main reference:

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