Final Exam

Before submitting, copy and sign the following statement.
"*I confirm that this exam is entirely my own work and that I have complied with the rules of academic integrity.*"
Duration : 2 hours.

1. **[2/40]** Show that the group $(\mathbb{C}/\mathbb{R},+)$ is isomorphic to $(\mathbb{R},+)$.

2. **[2/40]** Give an example of two non-isomorphic groups of order 10 (and justify your choice).

3. **[4/40]** Is the map $f:\mathbb{R}[x]\to\mathbb{R},\ P(x)\mapsto P'(0)$ an homomorphism of rings ? Same question for the map
$$g:\mathbb{R}[x]\to M_2(\mathbb{R}),\quad P(x)\mapsto\begin{pmatrix}P(0) & P'(0)\\ 0 & P(0)\end{pmatrix}.$$

4. Let p prime and consider the set
$$G:=\left\{\begin{pmatrix}a & b\\ 0 & a^{-1}\end{pmatrix},\ a,b\in\mathbb{Z}/p\mathbb{Z},\ a\neq\bar{0}\right\}.$$
   (a) **[2/40]** Show that G is a subgroup of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$.
   (b) **[1/40]** Compute $|G|$ the order of G.
   (c) **[3/40]** Identify a Sylow p-subgroup of G and show that it is normal.

5. Let G be the group of motion in the plane, and let $E(2)$ be the subset of G generated by translations t_a and rotations ρ_θ. (Recall that any element of $E(2)$ can be written in the form $t_a\circ\rho_\theta$.)
   (a) **[3/40]** Show that $E(2)$ is a subgroup of G.
   (b) **[3/40]** Show that $[G:E(2)]=2$.

6. Let $n\geq 1$ and denote by $H_n=\{\overline{1},\cdots,\overline{n-1}\}$ the set of nonzero residues mod n equipped with the multiplication.
   (a) **[1/40]** For which values of n is $(H_n,\times)$ a group ?
   (b) **[2/40]** Using the division algorithm, find $u,v\in\mathbb{Z}$ such that $8u+29v=1$.
   (c) **[2/40]** Deduce the inverse of $\overline{8}$ in H_{29}.
   (d) **[3/40]** Solve $8x\equiv 9\mod 29$.

7. Let G be a group of order 26.
   (a) **[2/40]** Show that G admits a normal subgroup H of order 13.
   (b) **[2/40]** Explain why H is cyclic spanned by some element x and show that $K=G/H=<y>$, with $y^2=1$.
   (c) **[2/40]** Show that there exists $k\in\{0,1,\cdots,12\}$ such that $yxy^{-1}=x^k$.
   (d) **[2/40]** Show that $x=x^{k^2}$.
   Indication : write $x=y^2x$ and use the fact that $yx=x^ky$.
   (e) **[2/40]** Deduce that $k=1$ or $k=12$.
   Indication : remember that $k^2-1=(k-1)(k+1)$.
   (f) **[2/40]** In the case where $k=12$, show that $(xy)^2=1$ and deduce that $G\simeq D_{13}$.
   (g) BONUS **[+4]** : in the case where $k=1$, show that $G\simeq C_{26}$, the cyclic group of order 26.