Midterm Exam

Before submitting, copy and sign the following statement.
"*I confirm that this exam is entirely my own work and that I have complied with the rules of academic integrity.*"

1. **[5/35]** Let $G = \{(x, y, z) \in \mathbb{Z}^3,\ x + y + z = 0\}$. Show that G is a subgroup of $(\mathbb{Z}^3, +)$ isomorphic to $\mathbb{Z}^2$.

2. **[5/35]** Over $\mathbb{R}$, consider the binary product given by $x * y = x + y - xy,\ \forall x, y \in \mathbb{R}$.
   (a) Show that $*$ is an associative commutative product that admits an identity element.
   (b) **(Bonus /3)** Explain why $(\mathbb{R}, *)$ is not a group.

3. Let $k \geq 0$ and let G be an Abelian group of order $|G| = 2k + 1$.
   (a) **[2/35]** For any $x \in G$, compute $(x^{k+1})^2$.
   (b) **[1/35]** Show that the map $f : G \to G,\ x \mapsto x^2$ is a homomorphism.
   (c) **[2/35]** Show that f is bijective, and hence an isomorphism.

4. **[5/35]** Consider a homomorphism of groups $f : (\mathbb{Q}, +) \to (\mathbb{Z}, +)$.
   Let $p \in \mathbb{Z}$ and $q \in \mathbb{Z} \setminus \{0\}$. Show that $f(p) = pf(1)$ and that $f(1) = qf(\frac{1}{q})$.

5. Consider $\sigma = (135)(24)(1267)(34) \in S_7$.
   (a) **[2/35]** Write σ as a product of disjoint cycles.
   (b) **[1/35]** Compute the signature of σ.
   (c) **[2/35]** Compute σ^{300}.

6. **[5/35]** In this question, we denote by $\mathbb{R}^*$ the multiplicative group of nonzero real numbers. Consider

$$G = \{f : \mathbb{R} \to \mathbb{R},\ x \mapsto ax + b,\ (a, b) \in \mathbb{R}^* \times \mathbb{R}\}; \quad H = \{f : \mathbb{R} \to \mathbb{R},\ x \mapsto x + b,\ b \in \mathbb{R}\}.$$

   Show that H is a normal subgroup of G.

7. **[5/35]** Consider $H = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, a \in \mathbb{R} \right\}$. The subset H is a subgroup of $\mathrm{SL}_2(\mathbb{R})$.
   Show that H is not normal in $\mathrm{SL}_2(\mathbb{R})$.
   Hint : you may want to use the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{R})$.