

### Midterm Exam - solutions

1. [5/35] Let  $G = \{(x, y, z) \in \mathbb{Z}^3, x + y + z = 0\}$ . Show that  $G$  is a subgroup of  $(\mathbb{Z}^3, +)$  isomorphic to  $\mathbb{Z}^2$ .

**Solution.** Let  $(x, y, z) \in G, (x', y', z') \in G$ .

- We have  $(x, y, z) + (x', y', z') = (x + x', y + y', z + z')$ . It follows that

$$x + x' + y + y' + z + z' = \underbrace{x + y + z}_{=0} + \underbrace{x' + y' + z'}_{=0} = 0.$$

Therefore,  $(x, y, z) + (x', y', z') \in G$ .

- For any  $(x, y, z) \in G$ , we have  $(x, y, z) + (0, 0, 0) = (x, y, z)$ , and obviously  $(0, 0, 0) \in G$ . It follows that  $(0, 0, 0)$  is the identity element of  $G$ .
- For any  $(x, y, z) \in G$ , we have that  $(-x, -y, -z) \in G$  and  $(x, y, z) + (-x, -y, -z) = (0, 0, 0)$ . Thus the inverse of  $(x, y, z)$  is given by  $(-x, -y, -z)$ . It follows that  $G$  is a subgroup of  $(\mathbb{Z}^3, +)$ .
- Let us show that  $G$  is isomorphic to  $\mathbb{Z}^2$ . Consider the map  $\varphi : \mathbb{Z}^2 \rightarrow G, (x, y) \mapsto (x, y, -x - y)$ . The map  $\varphi$  is an homomorphism of groups. Indeed, for  $x, x', y, y' \in \mathbb{Z}$ , we have

$$\begin{aligned} \varphi((x + y) + (x' + y')) &= \varphi(x + x' + y + y') = (x + x', y + y', -x - x' - y - y') \\ &= (x, y, -x - y) + (x', y', -x' - y') = \varphi(x, y) + \varphi(x', y'). \end{aligned}$$

We have

$$\ker(\varphi) = \{(x, y) \in \mathbb{Z}^2, \varphi(x, y) = (0, 0, 0)\} = \{(x, y) \in \mathbb{Z}^2, (x, y, -x - y) = (0, 0, 0)\} = \{(0, 0)\}.$$

Thus  $\varphi$  is injective. Moreover,  $\varphi$  is surjective, since  $(x, y, z) \in G \implies z = -x - y$ , because  $x + y + z = 0$ . It follows that  $\varphi$  is an isomorphism.

2. [5/35] Over  $\mathbb{R}$ , consider the binary product given by  $x * y = x + y - xy, \forall x, y \in \mathbb{R}$ .
- Show that  $*$  is an associative commutative product that admits an identity element.
  - (**Bonus /3**) Explain why  $(\mathbb{R}, *)$  is not a group.

**Solution.**

- The product  $*$  is clearly commutative since  $x * y = x + y - xy = y * x$ . Moreover, the identity for the product  $*$  is 0, since  $x * 0 = x, \forall x \in \mathbb{R}$ . Let us check associativity. Let  $x, y, z \in \mathbb{R}$ . We have

$$\begin{aligned} (x * y) * z &= x * y + z - (x * y)z \\ &= x + y - xy + z - xz - yz + xyz; \\ x * (y * z) &= x + (y * z) - x(y * z) \\ &= x + y + z - yz - xy - xz + xyz. \end{aligned}$$

it follows that  $(x * y) * z = x * (y * z)$  and the product  $*$  is associative.

- $(\mathbb{R}, *)$  is not a group because the element 1 does not admit an inverse. Indeed, let  $x \in \mathbb{R}$  such that  $x * 1 = 0$ . Then we would have  $1 + x - x = 0$  and therefore  $1 = 0$ , which is impossible.

3. Let  $k \geq 0$  and let  $G$  be an Abelian group of order  $|G| = 2k + 1$ .

- [2/35] For any  $x \in G$ , compute  $(x^{k+1})^2$ .
- [1/35] Show that the map  $f : G \rightarrow G, x \mapsto x^2$  is a homomorphism.
- [2/35] Show that  $f$  is bijective, and hence an isomorphism.

**Solution.**

- Let  $x \in G$ . Since  $G$  is a finite group of order  $2k + 1$ , we have  $x^{2k+1} = 1$ . It follows that

$$(x^{k+1})^2 = x^{2k+2} = x^{2k+1}x = x.$$

- (b) Since  $G$  is Abelian, we have  $f(xy) = (xy)^2 = xyxy = x^2y^2 = f(x)f(y)$ . Thus,  $f$  is a homomorphism of groups (actually, an endomorphism of  $G$ ).
  - (c) Using Question (a), for all  $x \in G$ , we have  $x = f(x^{k+1})$ . It follows that  $f$  is surjective. Moreover, since  $G$  is finite, the endomorphism  $f$  is also injective. This follows from the counting formula  $|G| = |\ker(f)| \times |\text{im}(f)|$  we have seen in Recitation, which is a corollary of the first isomorphism theorem. Thus,  $f$  is bijective and hence an isomorphism of groups.
4. [5/35] Consider a homomorphism of groups  $f : (\mathbb{Q}, +) \rightarrow (\mathbb{Z}, +)$ .

Let  $p \in \mathbb{Z}$  and  $q \in \mathbb{Z} \setminus \{0\}$ . Show that  $f(p) = pf(1)$  and that  $f(1) = qf(\frac{1}{q})$ .

**Solution.** Let  $p \in \mathbb{Z}$  and  $q \in \mathbb{Z} \setminus \{0\}$ . Since  $f$  is a homomorphism of groups, we have for  $p \geq 0$  :

$$f(p) = f(\underbrace{1 + 1 + \dots + 1}_{p \text{ times}}) = \underbrace{f(1) + f(1) + \dots + f(1)}_{p \text{ times}} = pf(1).$$

For  $p < 0$ , we notice that  $f(p) = -f(-p)$  and the same computation works. Moreover, we have  $q\frac{1}{q} = 1$ , and it follows that

$$f(1) = f(\underbrace{\frac{1}{q} + \frac{1}{q} + \dots + \frac{1}{q}}_{q \text{ times}}) = \underbrace{f(\frac{1}{q}) + f(\frac{1}{q}) + \dots + f(\frac{1}{q})}_{q \text{ times}} = qf(\frac{1}{q}).$$

If  $q$  is negative, the same trick as before also works.

*Remark.* The above computation shows that  $f$  is the trivial homomorphism. Indeed, we have shown that  $\frac{f(1)}{q} = f(\frac{1}{q}) \in \mathbb{Z}$  for all  $q \in \mathbb{Z} \setminus \{0\}$ . Thus, it follows that  $f(1) = 0$ . Therefore,  $f(p) = 0$  for any  $p \in \mathbb{Z}$ , and it follows that  $f \equiv 0$ .

5. Consider  $\sigma = (135)(24)(1267)(34) \in S_7$ .
- (a) [2/35] Write  $\sigma$  as a product of disjoint cycles.  $\sigma = (145)(2673)$ .
  - (b) [1/35] Compute the signature of  $\sigma$ .  $\varepsilon(\sigma) = \varepsilon(145)\varepsilon(2673) = (+1)(-1) = -1$ .
  - (c) [2/35] Compute  $\sigma^{300}$ . We have  $300 = 25 \times 12$ , so  $\sigma^{300} = (145)^{300}(2673)^{300} = \text{id}$ . Here it should be mentioned that cycles with disjoint support commute.

6. [5/35] In this question, we denote by  $\mathbb{R}^*$  the multiplicative group of nonzero real numbers. Consider

$$G = \{f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto ax + b, (a, b) \in \mathbb{R}^* \times \mathbb{R}\}; \quad H = \{f : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto x + b, b \in \mathbb{R}\}.$$

Show that  $H$  is a normal subgroup of  $G$ .

**Solution.** Consider the map  $\varphi : G \rightarrow \mathbb{R}^*$ ,  $f(x) = ax + b \mapsto a \in \mathbb{R}^*$ . Let  $f(x) = ax + b$  and  $g(x) = cx + d$ . We have  $f \circ g(x) = acx + ad + b$ . Thus,

$$\varphi(f \circ g) = ac = \varphi(f)\varphi(g).$$

It follows that  $\varphi$  is a homomorphism of groups. Moreover, we have that

$$\ker(\varphi) = \{f(x) = ax + b, a = 1\} = H.$$

It follows that  $H$  is the kernel of a homomorphism of groups, thus it is a normal subgroup of  $G$ .

7. [5/35] Consider  $H = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, a \in \mathbb{R} \right\}$ . The subset  $H$  is a subgroup of  $\text{SL}_2(\mathbb{R})$ .

Show that  $H$  is not normal in  $\text{SL}_2(\mathbb{R})$ .

*Hint : you may want to use the matrix  $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \text{SL}_2(\mathbb{R})$ .*

**Solution.** The subgroup  $H$  is normal in  $G$  if and only if  $ghg^{-1} \in H$ ,  $\forall g \in G$ ,  $\forall h \in H$ . Take  $g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ . We have  $g^{-1} = -g$ . Let  $h = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in H$ . We have

$$ghg^{-1} = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} \notin H.$$

It follows that  $H$  is not normal in  $G$ .