

Solutions

Problems 5, exercise 1

Let G be a group. For any $g \in G$, consider the map $\varphi_g : x \mapsto gxg^{-1}$. We denote by

$$\text{Int}(G) := \{\varphi_g, g \in G\}.$$

1. Show that $\text{Int}(G)$ is a normal subgroup of $\text{Aut}(G)$.
2. Consider the map $f : G \rightarrow \text{Int}(G), g \mapsto \varphi_g$. Show that f is a group homomorphism and compute its kernel.
3. Show that $G/Z(G) \cong \text{Int}(G)$.

Solution.

1. Let $g \in G$ and $u \in \text{Aut}(G)$. First, one need to check that φ_g is a morphism of groups (easy) and that its bijective (also easy ; its inverse is $\varphi_{g^{-1}}$). Next, we will show that $u\varphi_g u^{-1} \in \text{Int}(G)$. Let $x \in G$. We have

$$u\varphi_g u^{-1} = u(gu^{-1}(x)g^{-1}) = u(g)u \circ u^{-1}(x)u(g^{-1}) = u(g)xu(g^{-1}).$$

Thus, it follows that $u\varphi_g u^{-1} = \varphi_{u(g)} \in \text{Int}(G)$.

2. Let $g, h, x \in G$. We have

$$f(gh)(x) = \varphi_{gh}(x) = ghx(gh)^{-1} = g(hxh^{-1})g^{-1} = \varphi_g \circ \varphi_h(x) = f(g) \circ f(h)(x).$$

Thus, f is a morphism of groups.

Suppose that $g \in \ker(f) = \{x \in G, \varphi_x = \text{id} = \varphi_e\}$. We have

$$f(g) = \text{id} \iff gxg^{-1} = x \forall x \in G \iff gx = xg \forall x \in G \iff g \in Z(G).$$

Thus, it follows that $\ker(f) = Z(G)$.

3. We apply the first isomorphism theorem. The morphism of groups f is clearly surjective and its kernel is equal to $Z(G)$.

Therefore, it exists an unique isomorphism of groups $\tilde{f} : G/Z(G) \rightarrow \text{Int}(G)$ such that $f = \tilde{f} \circ \pi$, where π denotes the projection $\pi : G \rightarrow G/Z(G)$. In other words, the following diagram commutes :

$$\begin{array}{ccc} G & \xrightarrow{f} & \text{Int}(G) \\ \pi \downarrow & \nearrow \exists! \tilde{f} & \\ G/Z(G) & & \end{array}$$

Problems 5, exercise 2

Let G, H be groups of finite order such that $|G| = m, |H| = n$ with $\text{gcd}(m, n) = 1$. Determine all the group homomorphisms $f : G \rightarrow H$.

Solution.

Let $f : G \rightarrow H$. We will show that $f(g) = e_H \forall g \in G$, that is, the only valid group homomorphism is the trivial one.

By the first isomorphism theorem, we have that $G/\ker(f) \cong \text{Im}(f)$. Indeed, we have the diagram

$$\begin{array}{ccc} G & \xrightarrow{f} & \text{Im}(f) \\ \pi \downarrow & \nearrow \exists! \tilde{f} & \\ G/\ker(f) & & \end{array}$$

From this isomorphism, we deduce the following counting formula :

$$\boxed{|G| = |\ker(f)| \times |\text{Im}(f)|}.$$

Thus, $|\text{Im}(f)|$ must divide $|G| = m$. But since $\text{Im}(f)$ is a subgroup of H , $|\text{Im}(f)|$ must also divide $|H| = n$. Since $\gcd(m, n) = 1$, we deduce that $\boxed{\text{Im}(f) = \{e_H\}}$. The conclusion follows.

Problems 5, exercise 3

Let G be a group. An element $x \in G$ is called torsion element if $\exists n > 0, x^n = e$. The group G is called *torsion group* if all its elements are torsion elements and *torsion free* if the only torsion element of G is e .

Let G_1 be a torsion group and G_2 a torsion free group. Find all group isomorphisms $G_1 \rightarrow G_2$.

Solution.

Let $f : G_1 \rightarrow G_2$ a group homomorphism. Let $x \in G_1$ and let $y = f(x) \in G_2$. Since G_1 is a torsion group, there exists $n \geq 1$ such that $x^n = e_1$ (where e_1 denotes the identity element of G_1). Therefore, we have

$$y^n = f(x)^n = f(x^n) = f(e_1) = e_2.$$

Thus, we have $y^n = e_2 \in G_2$. But G_2 is torsion free, therefore we have $y = e_2$. It follows that $\boxed{f(x) = e_2 \forall x \in G_1}$ and the only homomorphism $G_1 \rightarrow G_2$ is the trivial homomorphism.