

Problems 6

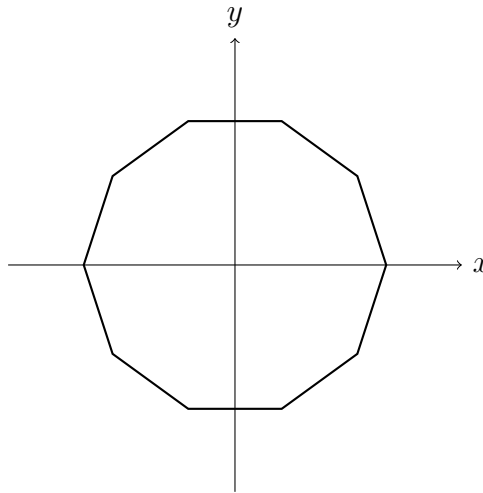
1. For $n \geq 1$, the Dihedral group is defined by

$$D_n = \{r^i s^j, r^n = s^2 = (rs)^2 = \text{id}\}.$$

- (a) Recall/explain the geometrical interpretation of D_n as the symmetry group of a regular planar n -gon (for $n \geq 3$).
- (b) Show that D_6 is isomorphic to $D_3 \times C_2$, where C_2 is the cyclic group of order 2.
Hint : use Artin's Prop. 2.11.4.

Solution.

- (a) Let $\theta = \frac{2\pi}{n}$. We denote by $r = \rho_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ the rotation of angle θ and by $s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ the reflexion with respect to the x -axis.



A regular planar 10-gon.

- (b) We recall Artin's Prop. 2.11.4. We need the item (d).

Proposition 1. *Let G be a group and let H, K be subgroups. Then,*

$$H \times K \cong G \iff H \cap K = \{e\}, H, K \text{ are normal in } G, \text{ and } HK = G.$$

We will apply this result with $H = D_3, K = C_2$.

- First, we have to show that D_3 and C_2 are subgroups of D_6 .

We have to find $\tilde{r} \in D_6$ and $\tilde{s} \in D_6$ such that $\tilde{r}^3 = \tilde{s}^2 = (\tilde{r}\tilde{s})^2 = \text{id}$. Let's focus on the rotations.

$$\begin{aligned} r^0 &= \text{id} \in D_3; \\ r &= \rho_{\frac{2\pi}{6}} \notin D_3; \\ r^2 &= \rho_{\frac{4\pi}{6}} = \rho_{\frac{2\pi}{3}} \in D_3; \\ r^3 &= \rho_{\pi} \notin D_3; \\ r^4 &= \rho_{\frac{8\pi}{6}} = \rho_{\frac{4\pi}{3}} \in D_3; \\ r^5 &= \rho_{\frac{10\pi}{6}} \notin D_3. \end{aligned}$$

Thus, for $k = 0, 1, 2$, we have $r^{2k} \in D_3$. Therefore, we can take $\boxed{\tilde{r} = r^2, \tilde{s} = 2}$. We have $\tilde{r}^3 = r^6 = \text{id}$ and $\tilde{s}^2 = s^2 = \text{id}$. It remains to show that $(\tilde{r}\tilde{s})^2 = \text{id}$. Note that

$$(rs)^2 = \text{id} \iff rs = sr^{-1}. \tag{1}$$

It follows that

$$(\tilde{r}\tilde{s})^2 = r^2sr^2s = r(\underbrace{rsrs}_{=\text{id}})r^{-1} = rr^{-1} = \text{id}.$$

Therefore, D_3 is a subgroup of D_6 and its generators are r^2 and s .

It remains to identify $K = C_2$. Recall that we want $D_3 \cap C_2 = \{\text{id}\}$, so we have to find a subgroup of D_6 of order 2 which does not intersect D_3 . For example, we can take $K = \langle r^3 \rangle$ the subgroup spanned by r^3 , which is clearly isomorphic to C_2 .

- Next, we show that H and K are normal in G . First, notice that from Eq. (1), we infer that $\boxed{r^i s = sr^{-i}, \forall i \geq 1}$. Let us show that $K = C_2$ is normal. We have

$$(r^i s^j)r^3(r^i s^j)^{-1} = (r^i s^j)r^3(s^{-j}r^{-i}).$$

But here, $j = 0, 1$. If $j = 0$, it's done. If $j = 1$, we have

$$(r^i s)r^3(sr^{-i}) = r^i s(r^3 s^{-1})r^{-i} = r^i s(sr^{-3})r^{-i} = r^i r^{-3}r^{-i} = r^{-3} = r^3 \in C_2.$$

It follows that C_2 is normal. For D_3 , we have $[D_6 : D_3] = 2$, thus D_3 is normal in D_6 .

- It remains to compute HK . We have

$$\begin{aligned} HK &= H \cup H\{r^3\} \\ &= \{r^{2k} s^j\} \cup \{r^{2k} s^j r^3\} \\ &= \{r^{2k} s^j\} \cup \{r^{2k-3} s^j\} = D_6. \end{aligned}$$

Therefore, we can apply the Proposition and it follows that $D_6 = D_3 \times C_2$.

2. Let G be a group and X be a set. The group G operates on X if there is a map

$$\begin{aligned} G \times X &\rightarrow X \\ (g, x) &\mapsto gx, \end{aligned}$$

such that $1x = x$ and $g(hx) = (gh)x$, $\forall g, h \in G, \forall x \in X$. Moreover, for all $x \in X$, we define

$$\text{Stab}(x) = \{g \in G, gx = x\}, \quad \text{Orb}(x) = \{y \in X, \exists g \in G, y = gx\}.$$

Show the following counting formula : $|G| = |\text{Stab}(x)| \times |\text{Orb}(x)|$, $\forall x \in X$.

Solution. Let X be a set, G a group acting on X and let $x \in X$. We consider the map

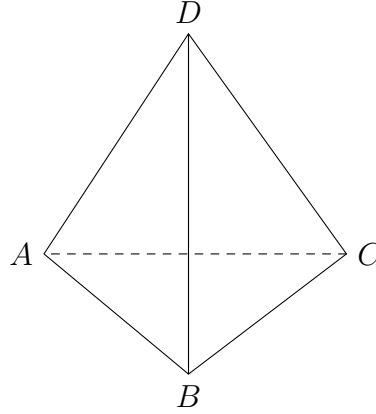
$$G/\text{Stab}(x) \rightarrow \text{Orb}(x), \quad \bar{g} \mapsto gx.$$

This map is well defined, because $\bar{g} = \bar{h} \implies gx = hx$; indeed, we have

$$\bar{g} = \bar{h} \iff g^{-1}h \in \text{Stab}(x) \iff g^{-1}hx = x \iff hx = gx.$$

Note that this map is not a group homomorphism, it's only a set map. The above chain of " \iff " show that the map is injective. Moreover, it is clearly surjective (be definition of the orbit). Therefore we have a set bijection between $G/\text{Stab}(x)$ and $\text{Orb}(x)$. The counting formula follows.

3. In this question, we will investigate the group of symmetries of a regular tetrahedron \mathcal{T} in \mathbb{R}^3 , that is, the group of transformations of the space that leave the regular tetrahedron invariant.
- Denote by G the set of symmetries of \mathcal{T} . Show that any $\varphi \in G$ leaves the set of vertices of \mathcal{T} invariant.
 - Show that G is a group equipped with a homomorphism $\chi : G \rightarrow S_4$.
 - Show that χ is a group isomorphism.



Solution.

- Let $\varphi \in G$. We denote the euclidian distance on \mathbb{R}^3 by d . We have

$$d(\varphi(x), \varphi(y)) = d(x, y), \quad \forall x, y \in \mathcal{T}.$$

Suppose that $d(A, B) = d(A, C) = \dots = 1$. Then, we have

$$d(\varphi(A), \varphi(B)) = d(\varphi(A), \varphi(C)) = \dots = 1.$$

That is, the points $\varphi(A), \varphi(B), \varphi(C), \varphi(D)$ form a regular tetrahedron. Since φ is a symmetry of \mathcal{T} , we have $\varphi(x) \in \mathcal{T}, \forall x \in \mathcal{T}$. It follows that $\varphi(x) \in \{A, B, C, D\}, \forall x \in \{A, B, C, D\}$.

- Let us show that G is a group.

- We have $\text{id} \in G$.
- Let $\varphi \in G$. Let us show that $\varphi^{-1} \in G$. Let $x, y \in \mathcal{T}$. We have

$$d(\varphi^{-1}(x), \varphi^{-1}(y)) = d(\varphi \circ \varphi^{-1}(x), \varphi \circ \varphi^{-1}(y)) = d(x, y).$$

Thus φ^{-1} is an isometry and obviously $\varphi^{-1}(x) \in \mathcal{T}$.

- Let $\varphi, \psi \in G, x, y \in \mathcal{T}$. We have

$$d(\varphi \circ \psi(x), \varphi \circ \psi(y)) = d(\varphi^{-1} \circ \varphi \circ \psi(x), \varphi^{-1} \circ \varphi \circ \psi(y)) = d(\psi(x), \psi(y)) = d(x, y).$$

We define a homomorphism of groups $\chi : G \rightarrow S_4$ as follows. We have $\varphi(\{A, B, C, D\}) \subset \{A, B, C, D\}$, each φ is a bijection of $\{A, B, C, D\}$, since it must preserve the distance. Indeed, there cannot exist $X \neq Y \in \{A, B, C, D\}$ such that $\varphi(X) = \varphi(Y)$, otherwise we would have

$$0 = d(\varphi(X), \varphi(Y)) = d(X, Y) = 1, \text{ a contradiction.}$$

Thus we define

$$\chi : G \rightarrow S_4, \quad \varphi \mapsto \sigma_\varphi, \quad \text{where } \sigma_\varphi(X) = \varphi(X), \quad \forall X \in \{A, B, C, D\}.$$

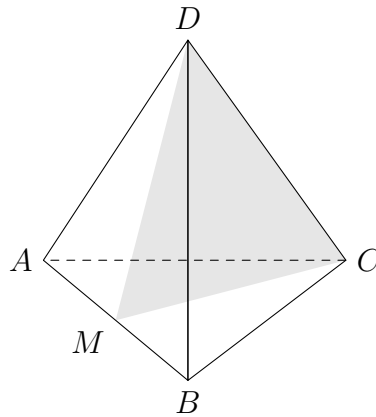
It is indeed a homomorphism :

$$\chi(\varphi \circ \psi)(X) = \sigma_{\varphi \circ \psi}(X) = \varphi \circ \psi(X) = \sigma_\varphi \circ \sigma_\psi(X) = \chi(\varphi) \circ \chi(\psi)(X).$$

(c) We show that the homomorphism χ is a bijection and hence an isomorphism. We have

$$\ker(\chi) = \{\varphi \in G, \sigma_\varphi = \text{id}\} = \{\varphi \in G, \varphi(X) = X, \forall X = A, B, C, D\} = \{\text{id}\}.$$

thus, χ is injective. Let us show that χ is also surjective. Recall that S_4 is generated by the transpositions; therefore, we will show that all transpositions are in $\text{im}(\chi)$. This is actually easy to see. Using the correspondence $A \mapsto 1, \dots, D \mapsto 4$, we will obtain the transposition (12) as the image by χ of some element of G . Let M be the middle point of the segment $[AB]$ and consider the plane P defined by the points C, D, M .



Let s_P be the symmetry with respect to the plane P . We have $s_P(A) = B$, $s_P(B) = A$, $s_P(C) = C$, $s_P(D) = D$. it's easy to see that $\chi(s_P) = (12)$. We can do the same trick for any transposition. It follows that χ is surjective, and hence an isomorphism.

Conclusion of the exercise : $G \cong S_4$.

4. In this question, we will investigate the group of symmetries of a cube \mathcal{C} in \mathbb{R}^3 , that is, the group of transformations of the space that leave the cube invariant. Denote this group by G . The group G is acting on the cube \mathcal{C} .
- (a) Let A be a vertex of the cube. Compute $\text{Stab}(A)$ and $\text{Orb}(A)$. Deduce that $|G| = 48$.
 - (b) Let \mathcal{D} be the set of big diagonals of the cube, that is, $\mathcal{D} = \{(IJ), X, Y \in \mathcal{C}, d(X, Y) = \sqrt{3}\}$. Denote by $S(\mathcal{D})$ the group of symmetries of \mathcal{D} . Show that the morphism

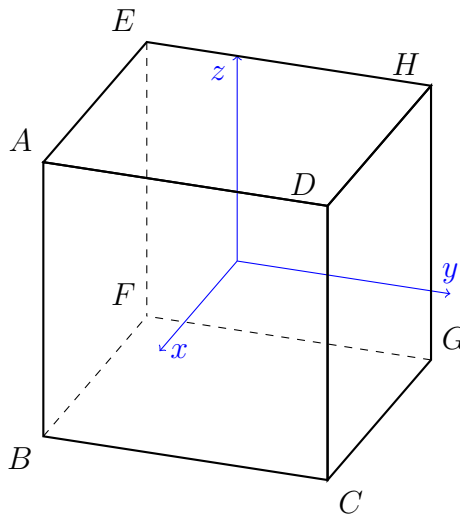
$$\psi : G \rightarrow S(\mathcal{D}), f \mapsto f|_{\mathcal{D}}$$

is surjective, but not injective. Show that $\ker(\psi) \cong \mathbb{Z}/2\mathbb{Z}$.

- (c) Consider the determinant $\det : G \rightarrow \{\pm 1\}$. Explain why it is a surjective group morphism. Denote by $G^+ = \ker(\det)$. Show that $|G^+| = |S_4|$ and deduce that $G^+ \cong S_4$.
- (d) Consider the map

$$\begin{aligned} \chi : G^+ \times \ker(\psi) &\rightarrow G, \\ (\sigma, \tau) &\mapsto \sigma \circ \tau. \end{aligned}$$

Show that χ is an isomorphism of groups. Deduce that $G \cong S_4 \times \mathbb{Z}/2\mathbb{Z}$.

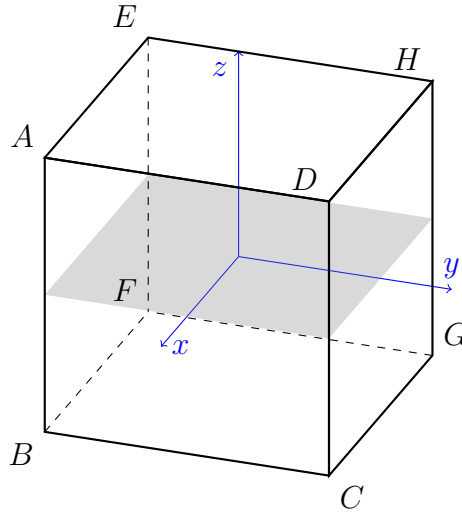


Solution.

- (a) We will show that $|\text{Orb}(A)| = 8$ and $|\text{Stab}(A)| = 6$. Recall that

$$\text{Orb}(A) = \{Y \in \mathcal{C}, \exists f \in G, Y = f(A)\}.$$

Denote by s_{xy}, s_{xz}, s_{yz} the reflexions with respect to the planes $(xy), (xz), (yz)$ respectively. By composition with these three reflexions, we can send A to any other point within the cube. For example, $G = s_{yz} \circ s_{xz} \circ s_{xy}(A)$. It follows that $\text{Orb}(A) = \{A, B, C, D, E, F, G, H\}$.



A reflexion wrt the plane (xy) sends A to B .

Next, we will show that $|\text{Stab}(A)| = 6$. Let $f \in \text{Stab}(A)$. We will investigate how f acts on the other vertices of \mathcal{C} . We suppose that the edges of the cube have length = 1. Since $A = f(A)$, we have

$$d(A, f(D)) = d(f(A), f(D)) = d(A, D) = 1.$$

Therefore, we have

$$f(D) \in \{X \in \mathcal{C}, d(A, X) = 1\} = \{B, D, E\}.$$

Similarly, we obtain that $f(B), f(E) \in \{B, D, E\}$. Let's check the different cases that may occur.

- The case where $f(D) = D$. If $f(B) = B$, then $f(E) = E$ and A, B, D, E are four non coplanar points that are fixed by f . It follows that $\boxed{f = \text{id}}$.
If $f(B) = E$, then $f(E) = B$ and f is a reflexion wrt the plane (ADG) : $\boxed{f = s_{ADG}}$.
- The case where $f(D) = E$. If $f(B) = B$, we have that $f(E) = D$ and f is a reflexion wrt the plane (ABG) : $\boxed{f = s_{ABG}}$.
If $f(B) = D$, we have that $f(E) = B$ and f is a rotation of angle $2\pi/3$ around the axis (AG) , let's call it r : $\boxed{f = r}$.
- The case where $f(D) = B$. If $f(B) = E$, we have $f(E) = D$ and f is a rotation of angle $4\pi/3$ around the axis (AG) : $\boxed{f = r^2}$.
If $f(B) = D$, we have that $f(E) = E$ and f is a reflexion wrt the plane (AEC) : $\boxed{f = s_{AEC}}$.

Therefore, we have exhausted all possible cases, and it follows that

$$\boxed{\text{Stab}(A) = \{\text{id}, r, r^2, s_{ADG}, s_{ABG}, s_{AEC}\}}.$$

It follows that $|\text{Stab}(A)| = 6$. Applying the counting formula of Ex. 2, we obtain

$$\boxed{|G| = |\text{Stab}(A)| \times |\text{Orb}(A)| = 6 \times 8 = 48}.$$

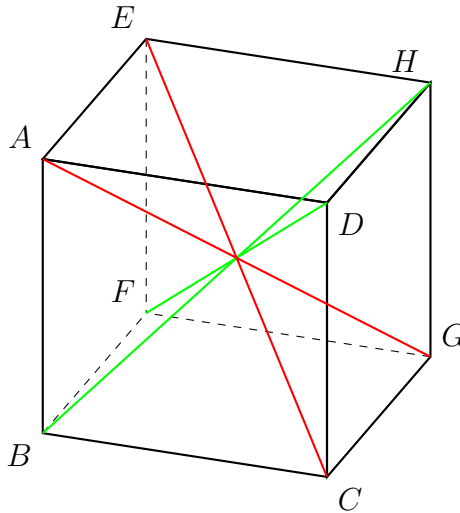
Remark. Note that if $f \in \text{Stab}(A)$, then we have $f(\{B, D, E\}) = \{B, D, E\}$. Thus, f is an isometry of the triangle BDE . It follows that $\text{Stab}(A) \cong D_3$.

(b) We have $\mathcal{D} = \{(AG), (BH), (CE), (DF)\}$. We consider the morphism

$$\psi : G \rightarrow S(\mathcal{D}) \cong S_4, f \mapsto f|_{\mathcal{D}}.$$

The morphism ψ is well defined, since for $X, Y \in \mathcal{D}$, we have

$$d(f(X), f(Y)) = d(X, Y) = \sqrt{3} \implies (f(X)f(Y)) \in \mathcal{D}.$$



- The morphism ψ is surjective. Any pair of diagonals defines a plane. For example, the diagonals (AG) and (CE) (the red ones) define a plane, let's call it P and denote by s_P the reflexion wrt the plane P . It's easy to see that s_P leaves the red diagonals invariant and exchanges the green ones. Thus, $\psi(s_P)$ is a transposition. We can build $\binom{4}{2} = 6$ such transpositions, thus we have them all. Since S_4 is generated by transpositions, it follows that ψ is surjective.
- The morphism ψ is not injective. It's easy to see that the symmetry s_0 wrt the origin 0 (the intersection of the four diagonals) is in the kernel of ψ . Indeed, we have $s_0(A) = G$, $s_0(E) = C$, etc so the set of diagonals is invariant. Thus, $\{\text{id}, s_0\} \subset \ker(\psi)$ and ψ is not injective.

Let us show that $\{\text{id}, s_0\} = \ker(\psi)$. By the first isomorphism theorem, we have $G/\ker(\psi) \cong S_4$, thus $|\ker(\psi)| = |G|/|\text{im}(\psi)| = 48/24 = 2$. The equality follows; therefore we have $\boxed{\ker(\psi) \cong \mathbb{Z}/2\mathbb{Z}}$.

(c) We consider $\det : G \rightarrow \{\pm 1\}$. It is clearly surjective, since $\det(\text{id}) = 1$ and $\det(s_0) = -1$. By the first isomorphism theorem again, we have $|G^+| = |G|/|\{\pm 1\}| = 48/2 = 24$. Then, we have

$$\boxed{|G^+| = 24 = |S_4|}.$$

Now, consider the restriction $\psi|_{G^+} : G^+ \rightarrow S_4$. It is injective, since s_0 is not in G^+ . It follows that we have an injective map $G^+ \rightarrow S_4$; since the orders of the groups are the same, it follows that $\boxed{G^+ \cong S_4}$.

(d) Consider the homomorphism

$$\begin{aligned}\chi : G^+ \times \ker(\psi) &\rightarrow G, \\ (\sigma, \tau) &\mapsto \sigma \circ \tau.\end{aligned}$$

Let's show that χ is an isomorphism. First, notice that s_0 is in the center of G ; it follows that χ is a homomorphism (to be done...). Moreover, χ is injective, because

$$\ker(\chi) = \{(\sigma, \tau), \sigma \circ \tau = \text{id}\} = \{(\sigma, \tau), \tau = \sigma^{-1}\} = \{(\text{id}, \text{id})\},$$

since $G^+ \cap \{\text{id}, s_0\} = \{\text{id}\}$. Finally, we have

$$|G^+ \times \ker(\psi)| = 24 \times 2 = 48 = |G|,$$

thus χ is an isomorphism.

Conclusion : $G \cong \mathbb{Z}/2\mathbb{Z} \times S_4$.
--