# Solutions 7

Let's start be recalling the Sylow Theorems.

**Theorem 1** (Cauchy)**.** *Let $G$ be a group of order $np$, with $p$ prime. Then, there exists a subgroup $H \subset G$ of order $p$.*

**Theorem 2** (Sylow)**.** *Let $G$ be a group. Suppose that $|G| = p^m n$, with $\gcd(n, p) = 1$.*

1. *There exists a subgroup $H \subset G$ of order $|H| = p^m$, called Sylow $p$-subgroup.*
2. *If there exists $K \subset G$ such that $|K| = p^a$ with $a \leq m$, then $K \subset g^{-1}Hg$ for some $g \in G$.*
3. *Denote by $\ell$ the number of Sylow $p$-subgroups of $G$. Then, $\ell | n$ and $\ell \equiv 1 \mod p$.*

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1. Let $G = S_4$. Explain why $G$ admits a subgroup of order 8 and exhibit one.

   **Solution.** We have that $|S_4| = 24 = 8 \times 3 = 2^3 \times 3$. By applying Sylow 1, there exists a subgroup $H$ of order 8. You can check that one appropriate choice is

   $$H = \langle (12); (34); (13)(24) \rangle$$

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2. Same question for $D_5$ and orders 2 and 5.

   **Solution.** We have $D_5 = 10 = 2 \times 5$. By Sylow 1, there exists a subgroup $H$ of order 5 and a subgroup $K$ of order 2. One can choose

   $$H = \langle \rho_{\frac{2\pi}{5}} \rangle, \quad K = \langle s \rangle,$$

   where $s$ is any reflexion ($s^2 = 1$).

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3. Show that all reflexions are conjugated in $D_n$.

   **Solution.** First, note that $|D_n| = 2n$ can always be written $2^k \times$ some odd integer, therefore we can apply the Sylow theorems. Let $K$ be a subgroup of order 2 of $D_n$. Then, there exists a reflexion $s$ such that $K = \langle s \rangle$. Let $s'$ be another reflexion and let $H = \langle s' \rangle$. We have $1 = a \leq m = 1$ in Sylow 2, thus it follows that $K$ and $H$ are conjugated. Therefore, all reflexions are conjugated in $D_n$.

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4. Let $p$ prime and $1 \leq n < p$. Let $G$ be a group of order $pn$ and $H$ a subgroup of order $p$. Show that $H$ is normal.

   **Solution.** We have $\gcd(p, n) = 1$, so $H$ is a Sylow $p$-subgroup. For any $g \in G$, we have that $g^{-1}Hg$ is also a Sylow $p$-subgroup. Therefore, to reach the conclusion, we have to show that $G$ admits only one Sylow $p$-subgroup.

   Let $\ell = |\{p - \text{Sylow}\}|$ the number of Sylow $p$-subgroups of $G$. By Sylow 3, we have that $\ell | n$ and $\ell \equiv 1 \mod p$. Suppose that $\ell \neq 1$. Then, we would have that $\ell \geq p + 1$, since $\ell \equiv 1 \mod p$. But this is impossible, since $\ell$ must divide $n < p$. Thus, $\ell = 1$.

   It follows that there is only one Sylow $p$-subgroup ; therefore, $H = g^{-1}Hg$ for all $g \in G$ and $H$ is normal.

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5. Recall that a group $G$ is called *simple* if its only normal subgroups are $\{1\}$ and $G$ itself. Let $G$ be a group of order 200. Show that $G$ cannot be simple.

   **Solution.** Write $200 = 8 \times 25 = 2^3 \times 5^2$. It follows that $G$ admits Sylow 2-subgroups and Sylow 5-subgroups. Denote by $\ell$ the number of Sylow 5-subgroups. By Sylow 3, we have that $\ell \equiv 1 \mod 5$ and $\ell$ divides $2^3 = 8$. It follows that $\ell = 1$. By applying the previous exercise, there is only one Sylow 5-subgroup, which is normal. Therefore, $G$ is not simple.

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6. Let $G$ be a group of order 10.
   (a) Show that there exists only one 5-Sylow $K =< x >$, with $x$ some element of order 5.
   (b) Let $H$ be a 2-Sylow. Explain why $H =< y >$, with $y$ some element of order 2 and show that $K \cap H = \{1\}$.
   (c) Show that there exists $1 \leq r \leq 4$ such that $yx = x^r y$.
   (d) Show that elements of the form $x^i y^j$ are distinct and deduce the group law of $G$.
   (e) Show that $r$ cannot be equal to $2, 3$.
   (f) Conclusion : show that $G \simeq C_5 \times C_2$ or $G \simeq D_5$.

   **Solution.**
   (a) The existence of such a $K$ follows from Sylow 1. By applying Sylow 3 (same argument as earlier), we have that there is only one Sylow 5-subgroup $K$, which is normal. Moreover, it is cyclic generated by some element $x$ of order 5.
   (b) The same argument as above shows that $H = \langle y \rangle$, with $y^2 = 1$. Since $K$ has an element of order 5 and $H$ is generated by an element of order 2, the intersection is trivial, since 2 doesn't divide 5.
   (c) Since $K$ is normal, we have that $yxy^{-1} \in K$. Since $K = \langle x \rangle$, there exists $1 \leq r \leq 4$ such that $yxy^{-1} = x^r$. The conclusion follows.
   (d) Let $i, j, k, l$ such that $x^i y^j = x^k y^l$. Then, we have
   $$x^{i-k} y^{j-l} = 1,$$
   thus $y^{j-l}$ is the inverse of $x^{i-k}$. It follows that $y^{j-l}$ and $x^{i-k}$ belong to $H \cap K = \{1\}$. Therefore, $y^{j-l} = x^{i-k} = 1$ and it follows that $i = k$ and $j = l$. We deduce that
   $$G = \{x^i y^j, \ 0 \leq i \leq 4, \ 0 \leq j \leq 1\}.$$
   Within $G$, we have the relations $x^5 = y^2 = 1$ and $yx = x^r y$. therefore, the group law of $G$ is entirely controlled by the value of $r$.
   (e) Suppose that $r = 2$. Then, we would have
   $$x = y^2 x = yyx = yx^2 y = x^4 y^2 = x^4.$$
   It follows that $x^3 = 1$, which is impossible, since $x$ has order 5.

   Suppose that $r = 3$. We can generalize the above computation. Actually, we have that
   $$x = y^2 x = \cdots = x^{r^2}.$$
   Therefore, $x^{r^2-1} = 1$, and $r^2 \equiv 1 \mod 5$, since $x$ has order 5. It follows that $r$ cannot be equal to 3.
   (f) Suppose that $r = 1$. In that case, the groups $K$ and $H$ commute by definition, and it follows that $G = C_5 \times C_2$, by applying Artin's Prop. 2.11.4., that we already used in Recitation 6. Suppose that $r = 4$. In that case, we recover the definition of the dihedral group $D_5$.